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Renormalization group flows for the second Z_5 parafermionic field theory.

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Abstract.

Using the renormalization group approach, the Coulomb gas and the coset techniques, the effect of slightly relevant perturbations is studied for the second parafermionic field theory with the symmetry Z_5 . New fixed points are found and classified.

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While the first series of parafermionic conformal field theories [1] is well studied and applied in various domains [2,3,4], the second parafermionic series, with the symmetry Z_N , has been developed fairly recently [5-8] and it still awaits its applications.

In the case of the first series, to a given N (of Z_N) is associated a single conformal theory. This is different for the second series: for a given Z_N , there exist an infinity of unitary conformal theories $Z_{N,p}$, with $p = N - 2 + k$, $k = 1, 2, 3, \dots \infty$. These theories correspond to degenerate representations of the corresponding parafermionic chiral algebra. They are much more rich in their content of physical fields, as compared to the theories of the first series. They are also much more complicated. But, on the other hand, the presence of the parameter p , for a given Z_N , opens a way to reliable perturbative studies. It allows in particular to study the renormalisation group flows in the space of these conformal theory models, under various perturbations.

In this paper we shall present results for a particular case of this problem: for the renormalization group flows of the $Z_{5,p}$ theories, being perturbed by two slightly relevant fields.

The details of the Z_5 parafermionic theory, the second one, could be found in [5]. The q charge of Z_5 takes values $q = 0, \pm 1, \pm 2$, so that in the Kac table of this theory one finds the Z_5 neutral fields, of $q = 0$, the $q = \pm 1$ and the $q = \pm 2$ doublets, and the Z_2 disorder fields. The symmetry of the theory is actually D_5 , which is made of Z_5 rotations and the Z_2 reflections in 5 different axes. These last symmetry elements amount to the charge conjugation symmetry: $q \rightarrow -q$.

We want to perturb by the Z_5 (D_5 in fact) neutral fields, in order to preserve this symmetry. Perturbatively well controlled domain of $Z_{5,p}$ theories is that of $p \gg 1$, giving a small parameter $\epsilon \sim 1/p$. This is similar to the original perturbative renormalization group treatment of minimal models for Virasoro algebra based conformal theory [9,10].

In this domain, i.e. for $p \gg 1$, one finds, in the lower part of the Kac table of Z_5 parafermionic theory, two Z_5 neutral fields which are slightly relevant and which close

by the operator algebra. They are:

$$S = \Phi_{(1,1|31)} \quad (1)$$

$$A = A_{-\frac{2}{5}} \Phi_{(1,1|1,3)} \quad (2)$$

The first one is a Z_5 singlet and the second is a parafermionic algebra descendant of a doublet $q = 1$ field. They both belong to the neutral $q = 0$ sector of Z_5 and they are both Virasoro algebra primaries.

Their labeling as S and A is just our shortened notations (in this paper) for these fields.

In general, the parafermionic algebra primaries of the second Z_5 conformal theory are labeled by double indices of two (α_+ and α_-) lattices of the B_2 classical Lie algebra [5]:

$$\Phi_{(n_1, n_2 | n'_1, n'_2)} \quad (3)$$

The first and second couples of indices correspond respectively to the α_+ and α_- B_2 lattices. α_+ and α_- are the usual Coulomb gas type parameters. For the $Z_{5,p}$ conformal theory they take the values:

$$\alpha_+ = \sqrt{\frac{p+2}{p}}, \quad \alpha_- = -\sqrt{\frac{p}{p+2}} \quad (4)$$

The formulas for the conformal dimensions of the fields (3) could be found in [5]. One could check that the dimensions of the fields S and A in (1) and (2) have the following values:

$$\Delta_S = \frac{5}{2}\alpha_-^2 - \frac{3}{2} = 1 - 5\epsilon \quad (5)$$

$$\Delta_A = \frac{3}{2}\alpha_-^2 - \frac{1}{2} = 1 - 3\epsilon \quad (6)$$

We have defined ϵ as follows:

$$\alpha_-^2 = \frac{p}{p+2} = 1 - \frac{2}{p+2} = 1 - 2\epsilon, \quad \epsilon = \frac{1}{p+2} \simeq \frac{1}{p} \quad (7)$$

Perturbing with the fields S and A corresponds to taking the action of the theory in the form:

$$\mathbf{A} = \mathbf{A}_0 + \frac{2g}{\pi} \int d^2x S(x) + \frac{2h}{\pi} \int d^2x A(x) \quad (8)$$

where g and h are the corresponding coupling constants; the additional factors $\frac{2}{\pi}$ are added to simplify the coefficients of the renormalization group equations which follow; \mathbf{A}_0 is assumed to be the action of the unperturbed $Z_{5,p}$ conformal theory.

It will be shown below that the operator algebra of the fields S and A is of the form:

$$S(x')S(x) = \frac{D_1}{|x' - x|^{4\Delta_S - 2\Delta_A}} A(x) + \dots \quad (9)$$

$$A(x')A(x) = \frac{D_2}{|x' - x|^{2\Delta_A}} A(x) + \dots \quad (10)$$

$$S(x')A(x) = \frac{D_1}{|x' - x|^{2\Delta_A}} S(x) + \dots \quad (11)$$

Only the fields which are relevant for the renormalization group flows are shown explicitly in the r.h.s. of the equations (9)-(11). For instance, the identity operator is not shown in the r.h.s. of (9) and (10) while it is naturally present there. The operator algebra constants in (9) and (11) should obviously be equal, as the two equations could be related to a single correlation function $\langle S(x_1)S(x_2)A(x_3) \rangle$.

Assuming the operator algebra expansions in (9)-(11), one finds, in a standard way, the following renormalization group equations for the couplings g and h :

$$\frac{dg}{d\xi} = 2 \cdot 5\epsilon \cdot g - 4D_1gh \quad (12)$$

$$\frac{dh}{d\xi} = 2 \cdot 3\epsilon \cdot h - 2D_2h^2 - 2D_1g^2 \quad (13)$$

These are up to (including) the first non-trivial order of the perturbations in g and h .

The problem now amounts to justifying the operator algebra expansions in (9) - (11) and to calculating the constants D_1 and D_2 .

The efficient method for calculating the operator product expansions and defining the corresponding coefficients, is that of the Coulomb gas technique.

Calculating directly the expansions of the products of the operators (1), (2) encounters a problem: the explicit form of the Coulomb gas representation for the Z_5 theory is not known. We shall get around this problem by using the coset representation for the (second) Z_5 theory and the related techniques. In particular, we shall generalize the methode developed in papers [11,12] for the $SU(2)$ cosets.

The second Z_5 theory, which we shall denote as $Z_5^{(2)}$, could be represented as the following coset construction [13]:

$$Z_{5,p}^{(2)} = \frac{SO_k(5) \times SO_2(5)}{SO_{k+2}(5)} \quad (14)$$

Here $SO_k(5)$ is the orthogonal affine algebra of level k ; $p = 3 + k$. This coset could be rewritten as follows:

$$Z_{5,p}^{(2)} \times \frac{SO_1(5) \times SO_1(5)}{SO_2(5)} = \frac{SO_k(5) \times SO_1(5)}{SO_{k+1}(5)} \times \frac{SO_{k+1}(5) \times SO_1(5)}{SO_{k+2}(5)} \quad (15)$$

The two coset factors in the r.h.s., as well as the additional coset factor in the l.h.s., correspond to the WB_2 theories [14]. For these theories the Coulomb gas representation is known. It is made of two bosonic fields, quantized with a background charge and the Ising model fields: Ψ (free fermion) and σ (spin operator) [14].

The equation (15) could be rewritten as

$$Z_{5,p}^{(2)} \times WB_{2,1} = WB_{2,k} \times WB_{2,k+1} \quad (16)$$

This equation relates the representations of the corresponding algebras. It could be reexpressed in terms of characters of representations, as is being usually done in the analyses of cosets. But this equation allows also to relate the conformal blocs of correlation functions. In doing so one relates the chiral (holomorphic) factors of physical operators. This later approach has been developped and analyzed in great detail in the papers [11,12], for the $SU(2)$ coset theories.

As it was said above, the chiral factor operators are related to the conformal bloc functions, not to the actual physical correlators. On the other hand, the coefficients of the operator algebra expansions are defined by the three point functions. These latter are factorizable, into holomorphic - antiholomorphic functions. So that, when the relation is established on the level of chiral factor operators, for the holomorphic three point functions, this relation could then be easily lifted to the relation for the physical correlation functions. Saying it differently, with the relations for the chiral factor operators one should be able to define the square roots of the physical operator algebra constants.

By matching the conformal dimensions of operators on the two sides of the coset equation (16) one finds the following decompositions for the operators S and A in (1) and (2) (their chiral factors in fact):

$$\Phi_{(11|31)}^{(Z_{5,p})} \times \Phi_{(11|11)}^{(WB_{2,1})} = \Phi_{(11|21)}^{(WB_{2,k})} \times \Phi_{(21|31)}^{(WB_{2,k+1})} \quad (17)$$

$$A_{-\frac{2}{5}} \Phi_{(1,1|1,3)}^{(Z_{5,p})} \times \Phi_{(11|11)}^{(WB_{2,1})} = a \Phi_{(11|11)}^{(WB_{2,k})} \times \Phi_{(11|13)}^{(WB_{2,k+1})} + b \Phi_{(11|13)}^{(WB_{2,k})} \times \Phi_{(13|33)}^{(WB_{2,k+1})} \quad (18)$$

The coefficients a and b in (18) are still to be determined. $\Phi_{(11|11)}^{(\dots)}$ are the identity operators. They could actually be dropped. But we shall keep them sometimes, when this makes the decomposition more explicit. The operator $\Phi_{(11|11)}^{(WB_{2,1})}$ in the l.h.s. of (17), (18) could be definitely suppressed.

By equations (17), (18), one observes that to decompose the products SS , AA , SA , as in eqs.(9)-(11), one needs to know the decompositions of products of the operators of $WB_{2,k}$ and $WB_{2,k+1}$ theories: $\Phi_{(11|21)}^{(WB_{2,k})} \times \Phi_{(11|21)}^{(WB_{2,k})}$, $\Phi_{(21|31)}^{(WB_{2,k+1})} \times \Phi_{(21|31)}^{(WB_{2,k+1})}$, etc.

These could be defined by using the Coulomb gas representation of the WB_2 theory. We shall give below the results of our analyses. The details of the calculations as well as more detailed analyses will be given in our next paper [15].

We have found the following values for the operator algebra constants of WB_2 :

$$\begin{aligned} D_{(11|13)(11|13)(11|13)} &= \frac{3\sqrt{2}}{\sqrt{5}}, \\ D_{(11|13)(13|13)(13|13)} &= \frac{3\sqrt{2}}{\sqrt{5}}\epsilon^2, \\ D_{(11|21)(11|21)(11|13)} &= \frac{2\sqrt{2}}{\sqrt{5}}, \\ D_{(21|31)(21|31)(11|13)} &= \frac{\sqrt{5}}{2\sqrt{2}}, \\ D_{(13|13)(13|13)(13|13)} &= 1, \\ D_{(21|31)(21|31)(13|13)} &= \frac{5}{8} \end{aligned} \quad (19)$$

These are the constants which are needed for our calculations. We give their values to the leading order in ϵ , which is sufficient for the renormalization group equations in (12), (13).

Using the above values of the WB_2 constants and performing the expansions for the products of operators in (17), (18), one finds that the products SS , AA , SA have the expansions of the form given in eqs.(9)-(11), with the constants D_1 , D_2 having the following values:

$$D_1 = \sqrt{\frac{5}{2}}, \quad D_2 = \frac{3}{\sqrt{10}} \quad (20)$$

In the process of these calculations one defines also the coefficients a and b in (18): $a = b = 1/\sqrt{2}$.

Some remarks are in order.

Doing the expansions of the products of $WB_{2,k}$ and $WB_{2,k+1}$ operators one does them:

1) with the square roots of the constants in (19);

2) one keeps in these expansions the "diagonal" cross-products only: the products of $WB_{2,k}$ and $WB_{2,k+1}$ operators which appear in coset relations for operators, due to eq.(16).

For operators, the relation (16) implies the decompositions of the following general form:

$$\begin{aligned} & \Phi_{(n_1, n_2 | n'_1, n'_2)}^{(Z_{5,p}^{(2)})} \times \Phi_{(s_1, s_2 | s'_1, s'_2)}^{(WB_{2,1})} \\ &= \sum_{l_1, l_2} a(l_1, l_2) \Phi_{(n_1 n_2 | l_1, l_2)}^{(WB_{2,k})} \times \Phi_{(l_1, l_2 | n'_1, n'_2)}^{(WB_{2,k+1})} \end{aligned} \quad (21)$$

The operators in this relation could be primaries or their descendants. The equations (17), (18) are two particular examples of eq.(21). These decompositions will be discussed in more detail in [15]. They generalize the corresponding relations for the $SU(2)$ cosets of [11,12].

The "diagonal" cross-products correspond to products (of $WB_{2,k}$ and $WB_{2,k+1}$ operators) of the type which appear in the r.h.s. of (21). The rest of possible cross-products have to be dropped when doing expansions.

The above features, square roots of constants and keeping the diagonal terms only, are due to the fact that we are dealing with the conformal bloc functions and not with

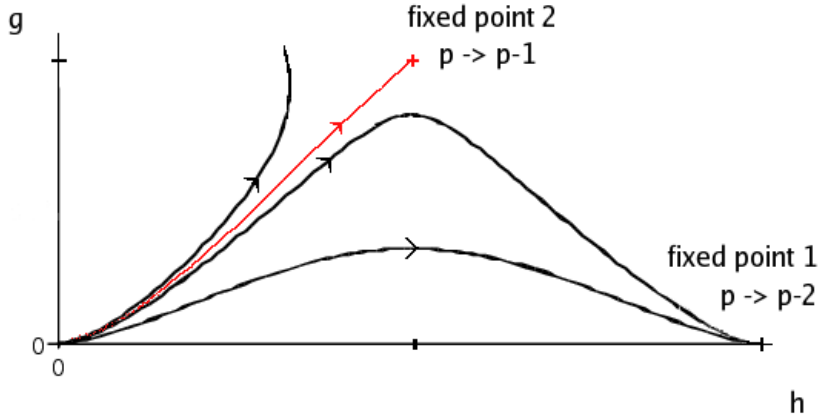
the actual physical correlators. These features are discussed in much detail in the paper [12].

Equally, the overall factors of the resulting expressions (the expressions which should correspond to the decomposed $Z_5^{(2)}$ operators) provide the square roots of the $Z_5^{(2)}$ structure constants, and not the constants themselves. In this way one obtains the square roots of the values of D_1 , D_2 in eq.(20).

Substituting now the values of D_1 , D_2 into the renormalization group equations (12), (13) and analysing them by the standard methods one obtains the following results.

The phase diagram of constants g and h contains:

- 1) the initial fixed point $g_0^* = h_0^* = 0$;
- 2) the fixed point on the h axis: $g_1^* = 0$, $h_1^* = \sqrt{10}\epsilon$;
- 3) two additional fixed points for non-vanishing values of the two couplings: $g_2^* = \sqrt{\frac{3}{2}}\epsilon$, $h_2^* = \frac{\sqrt{10}}{2}\epsilon$, and $g_3^* = -\sqrt{\frac{3}{2}}\epsilon$, $h_3^* = \frac{\sqrt{10}}{2}\epsilon$.



The renormalization group flows are shown in the figure. They are symmetrical with respect to $g \rightarrow -g$.

The value of the central charge at the point $g_1^* = 0$, $h_1^* = \sqrt{10}\epsilon$ agrees with that of the theory $Z_{5,p-2}^{(2)}$. This confirms the observation, made with the $SU(2)$ cosets [11,12] and, more generally, with the cosets for the simply laced algebras [16], that the perturbation of a coset theory caused by an appropriate operator drives p to $p - \Delta p$, Δp being equal to the shift parameter of the coset. In our case the shift parameter of the coset is equal to 2, eq.(14). Note that the algebra $B_2 \equiv SO(5)$ is not a simply laced one.

On the other hand, the appearance of two extra fixed points, (g_2^*, h_2^*) and (g_3^*, h_3^*) , is somewhat surprising. By the value of the central charge, the two critical points correspond to the theory $Z_{5,p-1}^{(2)}$. This assignment has further been verified by calculating the critical dimension of the operator $\Phi_{(1,n|1,n)}$ at these points.

We observe that such additional fixed points do not appear in the parafermionic model $Z_3^{(2)}$: the second Z_3 parafermionic theory with $\Delta_\Psi = 4/3$ [17]. This model could be realized by the $SU(2)$ cosets. Its perturbations, with two slightly relevant operators, have been analysed in [11,12].

Further analysis and discussions will be left for the paper [15].

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